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MATH 2055 Tutorial 5 (Oct 21)

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1. True or False.

   (a) $\{x_n\}$ converges $\iff$ all subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has a convergent subsequence $\{x_{n_{k_l}}\}$

   Solution: False

   $x_n = (-1)^n$ is counter example

   all subsequence of $\{x_n\}$ is bounded sequence, and hence the subsequence has a convergent subsequence by Bolzano Weierstrass Theorem

   but $\{x_n\}$ is divergent

   (b) If $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$, then $\{x_n\}$ converges.

   Solution: False

   $x_n = \sum_{i=1}^{n} \frac{1}{i}$ is counter example

   $\{x_n\}$ is increasing

$$
\begin{aligned}
x_{2^m} &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^m} \\
&= (\frac{1}{1}) + (\frac{1}{2}) + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \cdots + \frac{1}{8}) + \cdots (\frac{1}{2^r + 1} + \cdots \frac{1}{2^{r+1}}) + \cdots \\
&\quad + (\frac{1}{2^{m-1} + 1} + \cdots + \frac{1}{2^m}) \\
&\geq \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\
&= \frac{m+1}{2}
\end{aligned}
$$

   hence $\{x_n\}$ is unbounded and divergent

(c) If $f(\frac{1}{2^n})$ converge to $f(0)$, then f is continuous at 0

Solution: False

by definition, we should check all sequence which tends to 0 , not just a particular sequence

$$f(x) = \begin{cases} 0 & \text{if } x = \frac{1}{3^n} \text{ for some natural number n} \\ 1 & \text{otherwise} \end{cases}$$

is a counter example

it don't have right continuity

2. Prove that the following function is continuous

(a) $f(x) = r^x$ where r is positive real number

Solution:
$\forall x \in \mathbb{R}$,

case 1, $r \geq 1$

Recall that $\lim_{m\to\infty} r^{\frac{1}{m}} = 1$ (homework 2)

$\forall \epsilon > 0$, $\exists M_1$ such that for all $m > M_1$, $|r^{\frac{1}{m}} - 1| < \frac{\epsilon}{|r^x|}$

similarly, $\lim_{m\to\infty} r^{\frac{-1}{m}} = 1$

$\exists M_2$ such that for all $m > M_2$, $|r^{\frac{-1}{m}} - 1| < \frac{\epsilon}{|r^x|}$

for all sequence $\{x_n\}$ which tends to x,

$\exists N$, such that $\frac{-1}{max\{M_1,M_2\}+1} < x_n - x < \frac{1}{max\{M_1,M_2\}+1}$

because $r \geq 1$, $r^{\frac{-1}{max\{M_1,M_2\}+1}} \leq r^{-|x_n-x|} \leq r^{x_n-x} \leq r^{|x_n-x|} \leq r^{\frac{1}{max\{M_1,M_2\}+1}}$

$\implies 1 - \frac{\epsilon}{|r^x|} < r^{x_n-x} < 1 + \frac{\epsilon}{|r^x|}$

$|r^{x_n} - r^x| = |r^x||r^{x_n-x} - 1| < \epsilon$

$\therefore \{r^{x_n}\}$ tends to r^x, and hence f is continuous

case 2, for $r \leq 1$ we do similar things

(b) $f(x) = max\{g(x), h(x)\}$
where g, h are continuous function

Solution: take $x \in \mathbb{R}$,

Case 1, $h(x) \neq g(x)$, WLOG, we can assume $h(x) \geq g(x)$

$\forall \epsilon$ such that $\frac{h(x)-g(x)}{2} > \epsilon > 0$

because h is continuous, $\exists \delta_1$ such that $\forall y_1 \in (x - \delta_1, x + \delta_1)$, $|h(y_1) - h(x)| < \epsilon$

because g is continuous, $\exists \delta_2$ such that $\forall y_2 \in (x - \delta_2, x + \delta_2)$, $|g(y_2) - g(x)| < \epsilon$

let $\delta = max\{\delta_1, \delta_2\}$,

$\forall y \in (x - \delta, x + \delta)$,

$h(y) > h(x) - \frac{h(x)-g(x)}{2} = g(x) + \frac{h(x)-g(x)}{2} > g(y)$

$\therefore f(y) = h(y)$

$\implies |f(x) - f(y)| < \epsilon$

$\implies$ f is continuous at x

case 2, h(x) = g(x),

$\forall \epsilon > 0$

because h is continuous, $\exists \delta_1$ such that $\forall y_1 \in (x - \delta_1, x + \delta_1)$, $|h(y_1) - h(x)| < \epsilon$

because g is continuous, $\exists \delta_2$ such that $\forall y_2 \in (x - \delta_2, x + \delta_2)$, $|g(y_2) - g(x)| < \epsilon$

let $\delta = max\{\delta_1, \delta_2\}$,

$\forall y \in (x - \delta, x + \delta)$,

$$\begin{aligned} |f(x) - f(y)| &\leq max\{|f(x) - h(y)|, |f(x) - g(y)|\} \\ &= max\{|h(x) - h(y)|, |g(x) - g(y)|\} \\ &< \epsilon \end{aligned}$$

$\therefore$ f is continuous at x

(c) $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \sin \frac{1}{x} & \text{if } x \neq 0 \end{cases}$

Soliution:

only need to prove the continuity at 0

$\forall \epsilon > 0$, $\forall x \in (-\epsilon, \epsilon)$,

if $x \neq 0$,

$|f(x) - f(0)| = |x \sin \frac{1}{x}| \leq |x| < \epsilon$

if $x = 0$

$|f(x) - f(0)| = 0 < \epsilon$

$\therefore$ f is continuous at 0

3. given a sequence $\{x_n\}$, let $A = \{x | \exists$ subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ tends to x $\}$
Can A has uncountable infinitely many elements?

Solution:
set of all rational number $\mathbb{Q}$ are countable and hence you can list all rational number as a sequence

for any real number r , we can find a subsequence of the sequence above which tends to r

consider digital representation

4. Prove that all bounded sequence $\{x_n\}$ has a monotone subsequence.

   Solution:

   Fist, we define peak index

   m is a peak index for sequence $\{a_n\} \iff a_n \leq a_m \forall n \geq m$

   Case 1, if there are infinitely many peak index

   we can take $n_k =$ k-th peak index

   by definition, $a_k \geq a_{k+1}$ as k is peak index

   $\therefore \{a_{n_k}\}$ is decreasing sequence

   case 2, there are only finite peak index

   $\exists N$, such that there are no peak index greater than N

   take $n_1 = N + 1$,

   n_1 is not peak index ,

   $\therefore \exists n_2 > n_1$ such that $a_{n_2} > a_{n_1}$, also n_2 is not peak index

   recursively, we can take a increasing subsequence $\{a_{n_i}\}$

5. Given sequence of bounded sequence $\{a_{1,n}\}$, $\{a_{2,n}\}$, $\{a_{3,n}\}$, $\{a_{4,n}\}$, ...
   prove that there is a subsequence of natural number , say $\{n_k\}$, such that $\{a_{i,n_k}\}$ converge for all i

   Solution:

   idea: Subsequence of convergent sequence are convergent. we can try to apply Bolzano Weierstrass theorem iteratively such that the final subsequence "nearly" inside a convergent subsequence of each $\{a_{i,n}\}$

   $\because \{a_{1,m}\}$ is bounded, $\exists$ subsequence $\{m_{1,k}\}$ of $\{m\}$ such that $\{a_{1,m_{1,k}}\}$ converges take $n_1 = m_{1,1}$

$\{a_{2,m_{1,k}}\}$ is bounded, $\exists$ subsequence $\{m_{2,k}\}$ of $\{m_{1,k}\}$ such that $\{a_{2,m_{2,k}}\}$ converges

WLOG, we can assume $m_{2,1} > m_{1,1}$

take $n_2 = m_{2,1}$

Inductively, we can find a sequence n_k, such that a_{i,n_k} is a subsequence of $a_{i,m_{i,k}}$

$\Longrightarrow \{a_{i,n_k}\}$ converges for all i